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Directional measurement of technical efficiency of production: An axiomatic approach

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ABSTRACT

This contribution revisits the debate on the axiomatic properties satisfied by various radial versus non-radial measures of technical efficiency in production. This issue arises whenever isoquant and efficient subset of technology diverge and hence traditional radial measurement does not comply with Koopmans' definition of technical efficiency. This axiomatic approach to technical efficiency measurement is revisited within the framework of the more recently introduced directional distance function. This analysis provides the opportunity to define some new directional efficiency measures.

1. Introduction

In economics two distinct concepts of technical efficiency have emerged. The first concept, associated with Debreu (1951) and especially Farrell (1957), is related to the traditional radial efficiency measure. Focusing on input efficiency, it is defined as the minimal equiproportionate reduction in all inputs which still allows production of given outputs. This radial measure implicitly defines technical efficiency relative to the isoquant of technology. This radial input efficiency measure is the inverse of the input distance function that itself is dual to the cost function (Shephard, 1970). The second concept stems from the work of Koopmans (1951) who provided a definition of technical efficiency that focuses on the efficient subset of technology, but who refrained from defining a related efficiency measure. In his view a producer is technically efficient if an increase in any output or a decrease in any input requires a decrease in at least one other output, or an increase in at least one input. Thus, for each technology for which isoquant and efficient subset diverge, there is a potential conflict between both technical efficiency concepts.

To evaluate observations relative to this efficient subset, the theoretical literature has suggested a variety of non-radial efficiency indices as alternatives to the standard radial index which can conflict with Koopmans' definition of technical efficiency. A first article proposing an axiomatic approach to the problem is Färe and Lovell (1978) who suggested four properties that a measure of input efficiency should satisfy and proposed an alternative non-radial efficiency measure satisfying these axioms. This has led to a discussion in which these properties were scrutinized (e.g., Russell, 1985) and alternative non-radial efficiency measures were proposed (e.g., Zieschang, 1984).

More recently, a generalization of the Shephard (1970) distance function has been proposed known as the directional distance function to analyze both consumption and production theory. First, Luenberger (1992) defined the benefit function as a directional representation of preferences, thereby generalizing Shephard's (1970) input distance function defined in terms of a scalar output representing utility. Second, Luenberger (1995) introduced the shortage function which accounts for both input contractions and output improvements and which is dual to the profit function. Chambers et al. (1996) relabel this same function as a directional distance function and since then it is commonly known by that name. The latter authors analyze both the benefit function and the input-oriented directional distance function in some depth and extend the composition rules of McFadden (1978) to these new concepts.[1] For instance, a structural difference between directional compared to traditional distance functions is that the former have an additive structure while the latter are multiplicative in nature.

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[1] The same analysis could also apply to other general distance functions (for instance, McFadden's, 1978 gauge function or the generalized distance function of Chavas and Cox, 1999).

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However, while clearly generalizing existing distance functions, this directional distance function basically remains a functional representation of the isoquant of technology and hence the same problems alluded to before reappear in this new framework. Therefore, this contribution aims to revisit the issue of measuring efficiency with regard to the efficient subset in an input-oriented directional distance function perspective, to remain compatible with the original discussion that also took an input orientation.

Section 2 introduces the basic notation, the assumptions on technology, and some basic definitions like the relevant subsets of technology and the input directional distance function. Section 3 summarizes the original axiomatic literature on efficiency in terms of radial efficiency measures and redefines these same axioms for directional efficiency measures. Section 4 redefines the Färe and Lovell (1978), Zieschang (1984) and other efficiency measures in an input directional distance perspective and checks which of the new axioms are being satisfied. A final section concludes.

2. Notations, assumptions on technology, and basic concepts

Following the theoretical literature, the paper mainly concentrates on input efficiency. Technology describes all possibilities to transform input vectors $x=(x_1,\cdots,x_N)\in\mathbb{R}^N_+$ into output vectors $y=(y_1,\cdots,y_M)\in\mathbb{R}^M_+$. The production possibility set or technology T summarizes the set of all feasible input and output vectors: $T=\{(x,y)\in R^{N+M}_+ : x \text{ can produce } y\}$. Given our focus on input-oriented efficiency measurement, technology is represented by its input sets:

$$L(y)=\left\{x\in\mathbb{R}^N_+ : (x,y)\in T\right\}. \tag{2.1}$$

We assume the following properties on the input sets:

L.1: $L(0)=\mathbb{R}^N_+$ and $y\neq 0\Rightarrow 0\notin L(y)$;
L.2: Let $y_n\in\mathbb{R}^N_+, n\geq 0$ such that $\lim_{n\to\infty} y_n=+\infty$, then $\cap_{n\to\infty} L(y_n)=\emptyset$;
L.3: $L(y)$ is a closed set;
L.4: $x\in L(y)$ and $x'\geq x\Rightarrow x'\in L(y)$;
L.5: $x\in L(y)\Rightarrow \lambda x\in L(y)$ for all $\lambda\geq 1$;
L.6: $x\in L(y)\Rightarrow x+\alpha g\in L(y)$ for $g\in\mathbb{R}^N_+$ and all $\alpha\geq 0$.

Apart from the usual regularity assumptions (possibility of inaction, boundedness, and closedness), note that we do not impose any convexity assumption on the input sets. Three different input disposability assumptions can be alternatively imposed on the input sets. L.4 is the usual strong or free disposability of inputs assumption, while L.5 is its weak equivalent. The last assumption L.6 is termed g-input disposability: it is an input free disposability assumption in a specific direction g. This new assumption generalizes L.4: both assumptions are equivalent if $g=\mathbb{1}_N$, where $\mathbb{1}_N$ is the unit vector in $\mathbb{R}^N$. In the following, we mainly make use of L.4 and L.6. L.5 is mostly added for the sake of completeness, since it is related to the definition of technologies capable to detect congestion. In this contribution, congestion is simply interpreted as a special form of technical inefficiency.

Let us define the isoquant of an input set as

$$IsoqL(y)=\{x\in L(y) : \lambda x\notin L(y), \forall\lambda\in[0,1[\}. \tag{2.2}$$

The weak efficient subset is defined by

$$WEffL(y)=\{x\in L(y) : u<x\Rightarrow u\notin L(y)\}, \tag{2.3}$$

and the weak efficient subset in the direction of g of an input set as

$$WEff_gL(y)=\{x\in L(y) : x-\beta g\notin L(y), \forall\beta>0\}, \tag{2.4}$$

where $g\in\mathbb{R}^N_+$. Finally, the efficient subset of an input set is defined as

$$EffL(y)=\{x\in L(y) : u\leq x \text{ and } u\neq x\Rightarrow u\notin L(y)\}. \tag{2.5}$$

Notice that the weak efficient subset in the direction of g slightly differs from the usual notion of isoquant and weak efficient subset. A directional weak efficient subset uses the same direction g for all points of the input set, while the usual isoquant takes a direction following the input vector. Obviously, if $g\in\mathbb{R}^N_{++}$, then we have:

$$WEff_gL(y)=WEffL(y).$$

It is well known that $EffL(y)\subseteq WEffL(y)\subseteq IsoqL(y)\subseteq L(y)$. Obviously, one can also write that $EffL(y)\subseteq WEff_gL(y)\subseteq L(y)$.

Focusing on directional measures, let us denote $x+\mathbb{R}g$ the affine line in $\mathbb{R}^N$ defined by $x+\mathbb{R}g=\{x+\alpha g : \alpha\in\mathbb{R}\}$. We define the input directional distance function as follows: $D_i:\mathbb{R}^N_+\times\mathbb{R}^M_+\times\mathbb{R}^N_+\to\mathbb{R}\cup\{-\infty, +\infty\}$ as

$$D_i(x,y;g)=\begin{cases}\sup\{\beta : x-\beta g\in L(y)\} & \text{if } x+\mathbb{R}g\cap L(y)\neq\emptyset\\ -\infty & \text{if } x+\mathbb{R}g\cap L(y)=\emptyset.\end{cases} \tag{2.6}$$

This is related to the Debreu (1951)–Farrell (1957) input efficiency measure $(DF_i(x,y))$ as follows: $D_i(x,y;0)=1-DF_i(x,y)$ for $g=x$. Notice that different choices of direction vector may lead to different interpretations: e.g., choosing $g=x$ guarantees a proportional interpretation (see Briec, 1997).

This input directional distance function is dual to the cost function (see Chambers et al., 1996). Furthermore, it is well known that, under L.1 to L.4, this input directional distance function exhibits, among others, the following properties:

$$D_i(x+\alpha g,y;g)=D_i(x,y;g)+\alpha \tag{2.7}$$

and

$$D_i(x,y;g)\geq 0\Leftrightarrow x\in L(y). \tag{2.8}$$

The first property is a translation property. The last equivalence relation establishes that the input directional distance function completely characterizes technology (Chambers et al., 1996).

In the following section, we briefly present the results of different articles that are all aiming at defining the "best" input efficiency measure satisfying certain axioms following the seminal article of Färe and Lovell (1978). We also redefine these same axioms from the perspective of directional efficiency measurement.

3. An axiomatic approach to technical efficiency measurement

3.1. Axioms for radial efficiency measures

Färe and Lovell (1978) first suggested four properties that a measure of input efficiency should satisfy. These properties have been discussed in several contributions and particularly in Russell (1985, 1988). Let $E:\mathbb{R}^N_+\times\mathbb{R}^M_+\to\mathbb{R}_+\cup\{-\infty\}\cup\{+\infty\}$ be an efficiency measure. Formally, the desired properties were stated by Färe and Lovell (1978) as:

FL.1: If $x\in L(y), y>0$, then $E(x,y)=1\Leftrightarrow x\in EffL(y)$;
FL.2: If $x\in L(y)$ and $\lambda x\in L(y)$, then $E(\lambda x,y)=\lambda^{-1}E(x,y)$ for all $\lambda\in[\lambda^0,+\infty[$, where λ^0 satisfies $\lambda^0 x\in IsoqL(y)$;
FL.3: If $x\in L(y), y>0$, $u\geq x$ and $u\neq x$, then $E(x,y)>E(u,y)$;
FL.4: If $x\in L(y), y>0$ and $x\notin EffL(y)$, then $E(x,y)$ should compare to some $x^*\in EffL(y)$.

The first axiom requires that input vectors are efficient if and only if they belong to the efficient subset. The second axiom imposes homogeneity of degree minus one, e.g., a doubling of the input vector used by the inefficient observation halves its efficiency measure. The third axiom is strict negative monotonicity, i.e., increasing one input while holding all other inputs and outputs constant must lower the efficiency measure. The final axiom insists that inefficient input vectors be compared with respect to vectors in the efficient subset.

As pointed out by Russell (1985), the fourth property is mathematically ill defined. Furthermore, if mathematically interpreted in terms of a distance metric, then it can be shown that it is implied by the first three axioms. Consequently, this property is dropped in the remainder of the discussion. Furthermore, both Bol (1986) and Russell (1988) have shown that (FL.1), (FL.2) and (FL.3) are not compatible.

Färe and Lovell (1978) have shown that the radial Debreu (1951)–Farrell (1957) efficiency measure satisfies the homogeneity property (FL.2), but it fails the other three criteria for technologies satisfying L.1 to L.3 and L.5. They proposed to use another measure which they termed the Russell efficiency measure and which we refer to as the Färe–Lovell efficiency measure.[2] However, as noted by Färe et al. (1983), this measure does not satisfy the homogeneity property, but only a sub-homogeneity property of degree minus one. Moreover, Russell (1985) proves that it also fails to satisfy the strict monotonicity property (FL.3). Zieschang (1984) proposed a hybrid of the Debreu–Farrell and the Färe–Lovell efficiency measures. While this efficiency measure satisfies homogeneity of degree minus one, it does not always satisfy the monotonicity property. Finally, Färe (1975) defined an asymmetric input efficiency measure that looks for a proportional reduction of each input separately and then takes the minimum over these scalings (see also Färe et al., 1983). This asymmetric measure only satisfies (FL.1). Furthermore, it satisfies sub-homogeneity of degree minus one and weak monotonicity.

A more extensive overview of this whole literature can be found in Russell and Schworm (2009). This article includes a discussion of some additional axioms proposed in this literature (e.g., continuity in technologies and in input quantities: see Russell (1990)). Furthermore, it concentrates on the properties satisfied on a class of technologies generated by standard mathematical-programming methods rather than general technologies, which are the traditional focus in this literature.

All the previously cited articles deal with radial efficiency measures, but close to nothing has been done so far for directional efficiency measures.

3.2. Axioms for directional efficiency measures

We can now restate the desirable properties of efficiency measures in a directional framework by simply transposing the properties proposed by Färe and Lovell (1978), apart from the redundant axiom (FL.4). Let us define a directional input efficiency measure as the mapping $DE: \mathbb{R}_+^N \times \mathbb{R}_+^M \times \mathbb{R}_+^N \to \mathbb{R}_+ \cup \{-\infty, +\infty\}$ which must satisfy the following properties:

DFL.1: If $x \in L(y)$, then $DE(x,y;g) = 0 \Leftrightarrow x \in EffL(y)$;

DFL.2: If $x \in L(y)$, then $DE(x+\alpha g, y; g) = DE(x,y;g) + \alpha$ for all α such that $x+\alpha g \in L(y)$;

DFL.3: If $x \in L(y)$, $u \geq x$ and $u \neq x$, then $DE(x,y;g) < DE(u,y;g)$.

The first property is very similar to (FL.1). It states that a directional efficiency measure should identify all points of the efficient set. The only difference with the earlier approach is that efficiency is now characterized by a zero score. The second property transposes the homogeneity property (FL.2) into a translation property. The last monotonicity property requires that an input vector which uses at least more of one input should result in a less favorable efficiency measure.[3]

An obvious candidate for a "good" efficiency measure is the input directional distance function defined above. However, as established later in this section, it does not fulfill all the of directional Färe–Lovell requirements.

Let us first state the following proposition that is helpful to deal with the properties of the input directional distance function. In the following $\{e_n\}_{n=1,\ldots,N}$ denotes the canonical basis of $\mathbb{R}^N$.

Proposition 3.1. *Under L.1–L.4, we have for all* $y \in \mathbb{R}_+^M$:

$$EffL(y) = \bigcap_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y).$$

Proof. Assume that $x \in EffL(y)$. By definition, there exists no $u \leq x$ with $u \neq x$, such that $u \in L(y)$. In other words, there exists no $\beta > 0$ and $g \in \mathbb{R}_+^N$ such that $x - \beta g = u \in L(y)$. We deduce that $x \in \cap_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$. Therefore, $EffL(y) \subset \cap_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$. To show the converse, assume that $x \notin EffL(y)$. In such a case, there is some $u \in L(y)$ such that $u \leq x$ and $u \neq x$. Consequently, there is some $n \in \{1,\ldots,N\}$ such that $u_n < x_n$. Hence, there is some $\beta > 0$ such that $x - \beta e_n \in L(y)$. Thus, $x \notin WEff_{e_n} L(y)$, and consequently $x \notin \cap_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$. Thus, $\cap_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y) \subset EffL(y)$, and the converse is shown. □

This result has an immediate corollary:

Corollary 3.2. *Under L.1–L.4, we have for all* $y \in \mathbb{R}_+^M$:

$$EffL(y) = \bigcap_{n=1,\ldots,N} WEff_{e_n} L(y).$$

It is possible to go a bit further by connecting the isoquant and the directional weak efficient subset in the case when input factors are essential (i.e., there is a minimal level needed of all inputs to produce some outputs). Input factors are essential if for all $y \neq 0$ we have $L(y) \subset \mathbb{R}_{++}^N$.

Proposition 3.3. *Under L.1–L.4, and if the input factors are essential, then we have for all* $y \in \mathbb{R}_+^M \setminus \{0\}$:

$$IsoqL(y) = \bigcup_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y).$$

Proof. Assume that $x \in \cup_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$. In such a case, there is some $g \neq 0$ such that for all $\beta > 0$, $x - \beta g \notin L(y)$. Since the inputs are essential, we have $x > 0$. Therefore, there is no $\beta > 0$ such that $x - \beta x \in L(y)$. Consequently, there is no $\lambda \in [0,1[$ such that $\lambda x \in L(y)$ and we deduce that $x \in L(y)$. Therefore, $\cup_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y) \subset IsoqL(y)$. Conversely, if $x \in IsoqL(y)$, then $x \in WEff_x L(y)$. Hence, $x \in \cup_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$ and we deduce that $IsoqL(y) \subset \cup_{g \in \mathbb{R}_+^N \setminus \{0\}} WEff_g L(y)$ which ends the proof. □

The following corollary is then quite natural:

Proposition 3.4. *Under L.1–L.4, and if the input factors are essential, then we have for all* $y \in \mathbb{R}_+^M \setminus \{0\}$:

$$IsoqL(y) = \bigcup_{n=1,\ldots,N} WEff_{e_n} L(y).$$

[2] The name Russell efficiency measure is a misnomer. First, because the measure was first published in Färe and Lovell (1978). Second and foremost, Russell (1985, 1988) advocates the radial efficiency measure and even explicitly criticizes the use of this moniker (see Russell, 1998).

[3] However, this property might not be desirable under all assumptions imposed on technology. Indeed, we do not impose any strong disposability assumption in general. But, under a weak input disposability assumption, the technology may well be input congested with some input vectors exhibiting decreasing marginal productivity. In other words, an increase in the quantity of one input could lead to a lower output level (and sometimes to an infeasible production combination). In such a case, an increase in the congested input should not result in a decrease but rather in an increase of the efficiency measure since the output level y may not be producible any more with this increased input bundle. Therefore, (DFL.3) should only hold when inputs are strongly disposable.

As noted earlier (see Eq. (2.7)), a well-known property of the input directional distance function is that it exhibits the translation property (DFL.2). However, though by definition, $D_i(x,y;g)=0 \Leftrightarrow x\in WEff_gL(y)$, this does not imply that x belongs to $EffL(y)$. This can be shown by means of a simple counterexample as in Färe and Lovell (1978). Hence, the computation of $D_i(x,y;g)$ for a given direction g does not allow to conclude whether a production plan is efficient or not. Thus, (DFL.1) is not true in general. In fact, $x\in EffL(y)$ implies that $D_i(x,y;g)=0$, but the converse is only true in the special case when $EffL(y)=WEff_gL(y)$. Nonetheless, there exists a general equivalence that holds when the input directional distance function is null for all possible directions, as stated in the following proposition:

Proposition 3.5. *Under L.1–L.4, for all $(x,y)\in T$, $x\in EffL(y)$ if and only if $D_i(x,y;g)=0$ for all $g\in\mathbb{R}^N_+\backslash\{0\}$.*

Proof. We have $D_i(x,y;g)=0, \forall g\in\mathbb{R}^N_+ \Leftrightarrow x\in WEff_gL(y), \forall g\in\mathbb{R}^N_+ \Leftrightarrow x\in\cap_{g\in\mathbb{R}^+_N} WEff_gL(y)$. □

This proposition has an immediate corollary:

Corollary 3.6. *Under L.1–L.4, for all $(x,y)\in T$, $x\in EffL(y)$ if and only if $max_{n=1,\ldots,N}D_i(x,y;e_n)=0$.*

Hence, the traditional input directional distance function in a given direction cannot identify efficient points unless $WEff_gL(y)=EffL(y)$, which is a serious drawback for an efficiency measure.

Paralleling these results one can characterize the isoquant using the fact that the isoquant is the union of the directional isoquant.

Proposition 3.7. *Under L.1–L.4, if the input factors are essential, then for all $(x,y)\in T$ with $y\neq 0$, $x\in IsoqL(y)$ if and only if $D_i(x,y;g)=0$ for all $g\in\mathbb{R}^N_+\backslash\{0\}$.*

Hence, we can deduce the following corollary:

Corollary 3.8. *Under L.1–L.4, if the input factors are essential, then for all $(x,y)\in T$ with $y\neq 0$, $x\in IsoqL(y)$ if and only if $min_{n=1,\ldots,N}D_i(x,y;e_n)=0$.*

Now, if inputs are only weakly disposable then (DFL.3) fails to hold. However, as stated earlier, we do not want to impose this property of the efficiency measure when inputs are only weakly disposable. We rather want (DFL.3) to hold when they are strongly disposable (L.4). But, this is not the case as shown in the following example.

Example 3.9. *Let us define the input sets $L(y)=\{(x_1,x_2)\in\mathbb{R}^2_+ : min(x_1,x_2)\geq y\}$ for $y\in\mathbb{R}_+$. This technology satisfies L.1 to L.6. Now, for $g=(1,1)$, $x^*=(y,y)\in L(y)$ and $x=(y,y+\epsilon)\in L(y)$ with $\varepsilon>0$, we have $x\geq x^*$, $x\neq x^*$ and $DE(x^*,y;g)=DE(x,y;g)=0$ which contradicts (DFL.3).*

In fact, L.4 only implies a weak monotonicity condition which can be defined as:

DFL.3W: If $x\in L(y)$ and $u\geq x$, then $DE(x,y;g)\leq DE(u,y;g)$.

This is shown in the following proposition.

Proposition 3.10. *Under L.1–L.3, the input directional distance function satisfies (DFL.3W) if and only if inputs are strongly disposable.*

Proof. Chambers et al. (1996) have shown that the input directional distance function is weakly monotonic under a free disposal assumption. Conversely, if $u\geq x$ and the directional distance function is weakly monotonic, then $DE(x,y;g)\leq DE(u,y;g)$. However, $x\in L(y) \Leftrightarrow DE(x,y;g)\geq 0$. Consequently, $DE(u,y;g)\geq 0$ implies $u\in L(y)$. Hence, L.4 holds and the reciprocal is established. □

To summarize the results of this section succinctly, the input directional distance function satisfies (DFL.2), but neither (DFL.1) nor (DFL.3). However, it satisfies (DFL.3W) when the strong disposability assumption is imposed. One way to improve the properties of this distance function consists in imposing a more restrictive property, such as convexity of the input set. However, if we do not want to change the assumptions, then another type of directional efficiency measure is needed. Notice that the directional distance function can be easily computed when the production technology is non-parametric (see, for instance, Guironnet and Peypoch, 2007 and Barros et al., 2011).

4. Other directional efficiency measures

In this section, paralleling Färe and Lovell (1978), we characterize a Färe–Lovell input directional efficiency measure. Furthermore, we also define a directional version of the Zieschang (1984) input efficiency measure. Finally, an input directional version of the Färe (1975) asymmetric efficiency measure is established. The latter two definitions are – to the best of our knowledge – entirely new in the literature.

4.1. Efficient subset and Färe–Lovell directional efficiency measure

Denoting by ⊙ the Schur product of two vectors (element by element product), let us define the function $DFL:\mathbb{R}^N_+\times\mathbb{R}^M_+\times\mathbb{R}^N_+\to\mathbb{R}_+\cup\{-\infty,\infty\}$ as follows:

$$DFL(x,y;g)=\begin{cases}\sup_{\beta\in\mathbb{R}^{I(g)}_+}\left\{\frac{1}{|I(g)|}\sum_{n\in I(g)}\beta_n : x-\beta\odot g\in L(y)\right\} & \text{if } x\in L(y)\\ -\infty & \text{otherwise,}\end{cases} \quad (4.1)$$

where $I(g)$ is the index subset defined by $I(g)=\{n\in\{1,\ldots,N\}:g_n>0\}$ and $\mathbb{R}^{I(g)}_+=\{\sum_{n\in I(g)}\alpha_n e_n : \alpha_n\geq 0\}$. $|I(g)|$ stands for the cardinality of $I(g)$. Obviously, if $g\in\mathbb{R}^N_{++}$, $I(g)=\{1,\ldots,N\}$. Since this function is the directional equivalent of the Färe–Lovell input efficiency measure defined by Färe and Lovell (1978), we term it the Färe–Lovell input directional efficiency measure.[4] Note that the maximization is only computed when $x\in L(y)$. Otherwise, some of the β parameters could be negative and some positive at the same time. The Färe–Lovell input directional efficiency measure reduces to the input directional input directional in the direction g when $x\in L(y)$ and the constraint $\beta_n=\beta$ is added for all $n\in I(g)$, or in the case of a single input dimension. It is a directional efficiency measure but the direction is not preassigned as in the regular directional input distance function. Therefore, it can be rewritten as the maximization of the average of some other β coefficients weighted by the inverse of the directions by replacing $\beta_n g_n=\beta'_n$:

$$DFL(x,y;g)=\begin{cases}\max_{\beta'\in\mathbb{R}^{I(g)}_+}\left\{\frac{1}{|I(g)|}\sum_{n\in I(g)}\frac{\beta'_n}{g_n} : x-\beta'\in Ł(y)\right\} & \text{if } x\in L(y)\\ -\infty & \text{otherwise.}\end{cases}$$

One of the main drawbacks of the radial Färe–Lovell efficiency measure is that it does not satisfy the homogeneity property (FL.2) (see Färe et al., 1983) which is satisfied by the radial input efficiency measure. Instead, the Färe–Lovell efficiency measure is sub-homogeneous of degree minus one. The equivalent of the homogeneity property for a directional efficiency measure is the translation property. The Färe–Lovell input directional efficiency measure is no

[4] Briec (2000) was the first to define a slightly more general directional Färe–Lovell efficiency measure.

more convenient than its radial equivalent in that it does not satisfy the translation property (DFL.2) (see infra).

However, it can identify points in the efficient set $EffL(y)$ (DFL.1), while the directional distance function cannot. It also satisfies (DFL.3). These results are stated in the following proposition:

Proposition 4.1. *Under assumptions L.1 to L.4, if* $g \in \mathbb{R}^N_{++}$ *, then the Färe–Lovell input directional efficiency measure satisfies (DFL.1). Moreover, for all* $g \in \mathbb{R}^N_+ \backslash \{0\}$ *it satisfies (DFL.3) on* $\mathbb{R}^N_+ \times \mathbb{R}^M_+ \times \mathbb{R}^N_{++}$ *.*

Proof. If $g \in \mathbb{R}^N_{++}$, then $I(g) = \{1, \ldots, N\}$. Let us prove (DFL.1). Let $x \in L(y)$ such that $DFL(x, y; g) = 0$ and assume that $x \notin EffL(y)$. Therefore, there exists $u \leq x$ with $u \neq x$ such that $u \in L(y)$. Hence, since $g \in \mathbb{R}^N_{++}$, one can always find a vector α whose elements are $\alpha_n \geq 0, n = 1, \ldots, N$ with at least one n for which $\alpha_n > 0$ such that $u = x - \alpha \odot g \in L(y)$. This contradicts the fact that $DFL(x, u; g) = 0$. To show the converse, pick any $x \in EffL(y)$ and assume that $DFL(x, y; g) > 0$. From this last assumption, there exists $x - \beta \odot g \in L(y)$ with $\beta_n \geq 0$ for $n = 1, \ldots, N$ and at least one n for which $\beta_n > 0$. Now, $u = x - \beta \odot g$ is such that $u \leq x$, $u \neq x$ and $u \in L(y)$. Hence, we obtain a contradiction with the first assumption. Let us prove (DFL.3). Let $x \in L(y)$ and $u \geq x$ such that $u \neq x$. By hypothesis, since the strong disposal assumption holds, $\left\{\beta \in \mathbb{R}^{I(g)}_+ : x - \beta \odot g \in L(y)\right\} \subset \left\{\beta \in \mathbb{R}^{I(g)}_+ : u - \beta \odot g \in L(y)\right\}$, and the result follows. □

The next result shows that the Färe–Lovell input directional efficiency measure cannot exhibit the translation property.

Proposition 4.2. *Assume that* $g \in \mathbb{R}^N_{++}$ *. The Färe–Lovell input directional efficiency measure does not satisfy (DFL.2) for all technologies satisfying L.1–L.4.*

Proof. To prove this, we construct a technology that is not compatible with (DFL.2). Suppose that $N = 2$ and $M = 1$. Let $y \neq 0$ and suppose that:

$$V_0(y) = \left\{x \in \mathbb{R}^2_+ : x_1 \geq 1, x_2 \geq 4\right\}.$$

Let us denote $x_0 = (4, 4)$. Let $\epsilon > 0$ and let us define the point:

$$x_\epsilon = x_0 - \frac{\epsilon}{2} g$$

and consider the input set:

$$V_\epsilon(y) = \left\{x \in \mathbb{R}^2_+ : x \geq x_\epsilon\right\}.$$

Let us define the input set:

$$L(y) = V_0(y) \cup V_\epsilon(y).$$

Obviously, x_0 and x_ϵ are situated in $L(y)$ and are the only two efficient points in $L(y)$. Assume that $g > 0$, then it follows that

$$\begin{aligned} DFL(x_0, y) &= (1/2)\ max\left\{\frac{4-1}{g_1} + \frac{4-4}{g_2}, \frac{4-(4-\epsilon)}{g_1} + \frac{4-(4-\epsilon)}{g_2}\right\} \\ &= (1/2) max\left\{\frac{3}{g_1}, \frac{\epsilon}{g_1} + \frac{\epsilon}{g_2}\right\}. \end{aligned}$$

For ϵ sufficiently small, we have:

$$DFL(x_0, y) = \frac{3}{2g_1}.$$

However, if DFL is translation invariant, then we have:

$$DFL(x_0, y) = DFL(x_\epsilon + \epsilon g, y) = DFL(x_\epsilon, y) + \epsilon = \epsilon,$$

which is a contradiction. □

But, the Färe–Lovell input directional efficiency measure does not compare the input vector with the closest efficient point. Nor does it use a predetermined path to reach this point. Following a specific direction g, which can be the average input mix or keeping the input mix proportions unchanged, may have some economic meaning. But, following any direction does not have an evident economic sense. This is why we define another measure of efficiency which is very close to this Färe–Lovell one, but that is more economically meaningful.

Notice that so far we have assumed that the direction g has positive components. But, one can extend this approach to consider the situation where some inputs are fixed in the short run. In such a case, the null components of the direction vector correspond to the subvector of fixed inputs. Everything developed so far obviously remains true for all input directional efficiency measures developed in this contribution.

4.2. "Additive formulation" of the Zieschang measure and decomposition of the Färe–Lovell measure

The lack of translation property of the Färe–Lovell input directional efficiency measure implies that it can be useful to define an additive analog of the multiplicative Zieschang (1984) measure to explore a way out. The interest of the Zieschang measure comes from the fact that it preserves homogeneity. We define over $\mathbb{R}^M_+ \times \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \rightarrow \mathbb{R}_+ \cup \{-\infty\}$ the function

$$DZ(x, y; g) = \begin{cases} DFL(x - D_i(x, y; g)g, y; g) + D_i(x, y; g) & \text{if } x \in L(y) \\ -\infty & \text{otherwise.} \end{cases} \quad (4.2)$$

This Zieschang input directional efficiency measure proceeds in two steps. First, it reaches a weakly efficient point (in $WEff_g L(y)$). Second, it translates this last input vector to one element of the efficient subset. Hence, if $x \in EffL(y)$, then the Zieschang input directional efficiency measure equals the Färe-Lovell input directional efficiency measure. By contrast, when $x - D_i(x, y; g) \in EffL(y)$, then it equals the input directional distance function.

Proposition 4.3. *Under assumptions L.1 to L.4, if* $g \in \mathbb{R}^N_{++}$*, then the Zieschang input directional efficiency measure satisfies (DFL.1). Moreover, for all* $g \in \mathbb{R}^N_+ \backslash \{0\}$*, it satisfies (DFL.2).*

Proof. (DFL.1) and (DFL.2) are obvious by construction. □

Unfortunately, the Zieschang input directional efficiency measure fails to satisfy (DFL.3) and (DFL.3W).

Proposition 4.4. *Assume that* $g \in \mathbb{R}^N_{++}$ *. The Zieschang input directional efficiency measure does not satisfy (DFL.3W) for all technologies satisfying L.1–L.4.*

Proof. To prove this, we construct a technology that is not compatible with (DFL.2). Suppose that $N = 2$ and $M = 1$. Let $y \neq 0$ and suppose that

$$V_0(y) = \left\{x \in \mathbb{R}^2_+ : x_1 \geq 1, x_2 \geq 4\right\},$$

and

$$V_1(y) = \left\{x \in \mathbb{R}^2_+ : x_1 \geq 3, x_2 \geq 3\right\}.$$

Assume that $L(y) = V_0(y) \cup V_1(y)$. Let $\epsilon > 0$ and let us define the point

$$\bar{x}_0 = (3, 4) \text{ and } \quad x_\epsilon = (3 + \epsilon, 4).$$

We have obviously $x_\epsilon \geq x_0$. Since the general case ($g > 0$) can be deduced via a single change in the units of measurement, we suppose that $g = (1, 1)$. We have $D(x_0, y; g) = 0$ and $DFL(x_0, y; g) = 1$, consequently DZ

$(x_0,y;g)=1$. Moreover, $D(x_\epsilon,y;g)=\epsilon$ and $DFL(x_\epsilon-\epsilon g,y;g)=\frac{1-\epsilon}{2}$, consequently $DZ(x_\epsilon,y;g)=\frac{1+\epsilon}{2}$. Hence, for ϵ sufficiently small $DZ(x_\epsilon,y;g)<DZ(x_0,y;g)$, which contradicts (DFL.3W). □

Now, using the translation property (2.7), we obtain:

$$DZ(x,y;g)=DFL(x,y;g)-D_i(x,y;g)+D_i(x,y;g) \tag{4.3}$$

$$=DFL(x,y;g) \tag{4.4}$$

This formulation serves to show how to decompose the Zieschang input directional efficiency measure to compute it in two steps. The first step translates an input vector onto the frontier, and the second step computes the Färe–Lovell measure of the thus translated input.

Paralleling Bol (1986) one can prove a more general result: no directional efficiency measure can simultaneously satisfy these three requirements. The following example provides a technology for which an efficiency measure satisfying (DFL.1) and (DFL.2) cannot satisfy (DFL.3):

Example 4.5. *We consider a technology with 2 inputs and one output. Pick a vector $g=(1,1)$. For all $y\in\mathbb{R}_+$ we define $L(y)$ as follows:*

$$L(y)=\begin{cases} x_1+x_2\geq 2y & \text{for} \quad 0\leq x_1\leq y\\ x_2\geq y & \text{for} \quad y\leq x_1\leq y+2\\ x_1+x_2\geq 2+2y & \text{for} \quad 2+y\leq x_1\end{cases} \tag{4.5}$$

It is easy to see that this correspondence fulfills the basic axioms L1–L6. For $\mu\geq 0$ let us define $x^\mu=(y+\mu,y)$.

We have for $\mu\geq 2$:

$$\begin{aligned}\delta_\mu &= \max\{\delta : x^\mu-\delta(1,1)\in L(y)\} & \text{(a)}\\ &= \max\{y+\mu+y-\delta-\delta\geq 2+2y\} & \text{(b)}\\ &= \frac{\mu-2}{2}. & \text{(c)}\end{aligned} \tag{4.6}$$

By definition, $x^\mu-\delta_\mu(1,1)\in Eff(L(y))$ for $\mu>2$. By hypothesis, a distance function, DE, must satisfy

$$DE(x^\mu,y;1,1)=DE\left(x^\mu-\delta_\mu(1,1),y;1,1\right)+\delta_\mu$$

for $\mu>2$. Thus, if μ converges to 2 both $DE(x^\mu,y;1,1)$ and δ_μ converges to 0. On the other hand, $x^2=(y+2,y)\notin EffL(y)$. Hence, $DE(x^2,y;1,1)>0$. Thus, obviously a $\mu>2$ exists such that

$$DE(x^\mu,y;1,1)<DE\left(x^2,y;1,1\right) \tag{4.7}$$

and since $x^\mu\geq x^2$ and $x^\mu\neq x^2$, this is a contradiction of (DFL.3).

4.3. Asymmetric Färe input directional efficiency measure

Färe (1975) introduced an asymmetric input efficiency measure (see also Färe et al., 1983) defined by:

$$AF(x,y)=\begin{cases}\min_{n\in I(x)}\inf\{\lambda : (\mathbb{1}_N+(\lambda-1)e_n)\odot x\in L(y)\} & \text{if } x\in L(y)\\ +\infty & \text{otherwise.}\end{cases} \tag{4.8}$$

where $I(x)=\{n\in\{1,\dots,N\}:x_n>0\}$ and $\mathbb{1}_N$ is the unit vector of $\mathbb{R}^N$. This measure is based upon a proportional reduction of each input dimension separately in a first step. The asymmetric Färe measure is then calculated by taking the minimum value of the scores obtained from each input dimension. Clearly, this efficiency measure allows characterizing the efficient subset of the input set (i.e., it satisfies (FL.1)).

It is relatively easy to define an analogous measure in the context of the directional distance function. The asymmetric Färe input directional efficiency measure is the map $DAF:\mathbb{R}^N_+\times\mathbb{R}^M_+\times\mathbb{R}^N_+\to\mathbb{R}\cup\{-\infty\}$ defined by:

$$DAF(x,y;g)=\begin{cases}\max_{n\in I(g)}\sup\{\beta : x-\beta g_n e_n\in L(y)\} & \text{if } x\in L(y)\\ -\infty & \text{otherwise.}\end{cases} \tag{4.9}$$

It is shown below that, by construction, this measure indeed allows identifying a strongly efficient point on the frontier. Hence, it satisfies (DFL.1).

Proposition 4.6. Under assumptions L1–L4, if $g\in\mathbb{R}^N_{++}$, *then the asymmetric Färe input directional efficiency measure (DAF) satisfies (DFL.1) and (DFL.3).*

Proof. (a) We need to prove that if $x\in L(y)$, then $DAF(x,y;g)=0\Leftrightarrow x\in EffL(y)$. If $g\in\mathbb{R}^N_{++}$, then $I(g)=\{1,\dots,N\}$. Suppose that $x\in L(y)$ and assume there is some $u\in L(y)$ such that $u\leq x$ and $u\neq x$. In such a case, there is some $n\in\{1,\dots,N\}$ such that $u_n<x_n$. Therefore, $max_{n=1\dots N}\{\beta : x-\beta g_n e_n\in L(y)\}>0$ and consequently $DAF(x,y;g)>0$. Thus, $DAF(x,y;g)=0$ implies that $x\in EffL(y)$. Conversely, if $DAF(x,y;g)>0$, then $x\notin EffL(y)$, which ends the first part of the proof. (b) We need to prove that if $x\in L(y)$ and $u\geq x$ with $u\neq x$, then $DAF(x,y;g)<DAF(u,y;g)$. Suppose that $x\in L(y)$, $u\geq x$ and $u\neq x$. Since $u\geq x$, $D_i(u,y;g_ne_n)\geq D_i(x,y;g_ne_n)$ for each n. However, there is also some $n_0\in\{1,\dots,N\}$ such that $u_{n_0}>x_{n_0}$, and since $g\in\mathbb{R}^N_{++}$, this implies that $D_i(u,y;g_{n_0}e_{n_0})>D_i(x,y;g_{n_0}e_{n_0})$. Hence, $DAF(u,y;g)=max_{n=1\dots N}D_i(u,y;g_ne_n)>max_{n=1\dots N}D_i(x,y;g_ne_n)=DAF(x,y;g)$, which ends the proof. □

Note that the proof of the first part of the above proposition can also be established as an immediate consequence of Corollary 3.2. It is easy to check that this measure fails to satisfy (DFL.2).

5. Concluding comments

This contribution has summarized the possibilities to define efficiency measures complying with Koopmans' definition of technical efficiency in the framework of the traditional radial distance functions and of the rather recently introduced directional distance functions. After summarizing the original axiomatic literature on technical efficiency in terms of radial and non-radial efficiency measures, we redefine these same axioms in the context of the input directional efficiency measures.

Thereafter, we analyze the properties satisfied by the input directional distance function and the Färe–Lovell directional efficiency measure. Neither of these two measures turns out to simultaneously satisfy all of these newly defined properties. Furthermore, we also define an additive formulation of the Zieschang (1984) measure. Finally, we define a directional version of the asymmetric Färe (1975) efficiency measure. Again, both of these newly defined input directional measures of technical efficiency do not satisfy all of the new axioms. More generally, we prove that no input directional efficiency measure can satisfy all of the newly required properties.

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